Complexity upper bounds for JD4

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Computational Logic Seminar

Outline

Justification Logic - Syntax

- Constants: $c_i, \ i \in \mathbb{N}$
- Variables: $x_i, \ i \in \mathbb{N}$
- Justification Terms :
  - Constants and variables are terms
  - If t_1, t_2 are terms, so are

$$(t_1 \cdot t_2), \ (t_1 + t_2), \ (!t_1)$$

- $\cdot$ is called application, $+$ is called sum and ! proof checker.
- Sentence letters: $p_i, \ i \in \mathbb{N}$
- If p is a sentence letter, t is a term and ϕ_1, ϕ_2 are formulas, then so are

$$p, \ \bot, \ (\phi_1 \to \phi_2), \ (t : \phi_1)$$

$J_\emptyset$

- Finitely many schemes of classical propositional logic
- $s : (\phi \to \psi) \to (t : \phi \to s \cdot t : \psi)$ - Application Axiom
- $$s : \phi \to s + t : \phi$$
  $$s : \phi \to t + s : \phi$$
  - Monotonicity Axiom
- Modus Ponens Rule :

$$\frac{\phi \to \psi \qquad \phi}{\psi}$$

JD, JT, J4, JD4, LP

- $t : \phi \rightarrow \phi$ - Factivity Axiom
- $t : \phi \rightarrow !t : t : \phi$ - Positive introspection
- $t : \bot \rightarrow \bot$ - Consistency Axiom
- $\mathrm{JD}_\emptyset = \mathrm{J}_\emptyset +$ Consistency
- $\mathrm{JT}_\emptyset = \mathrm{J}_\emptyset +$ Factivity
- $\mathrm{J4}_\emptyset = \mathrm{J}_\emptyset +$ Positive introspection
- $\mathrm{JD4}_\emptyset = \mathrm{J}_\emptyset +$ Consistency + Positive introspection
- $\mathrm{LP}_\emptyset = \mathrm{J}_\emptyset +$ Factivity + Positive introspection

Constant Specification

- A constant specification for a justification logic JL is any set

$$\mathcal{CS} \subset \{c : A \mid c \text{ is a constant, } A \text{ an axiom of } JL\}$$

A c.s. is:

- axiomatically appropriate if each axiom is justified
- injective if every constant justifies at most one axiom
- schematic if every constant justifies a certain number of axiom schemes
- schematically injective if it is schematic and every constant justifies at most one scheme
- finite if it is a finite set
- almost schematic if it is the union of a schematic and a finite c.s.

More Inference Rules

R4$_{\mathcal{CS}}$

$$\overline{c : A},$$

where $c : A \in \mathcal{CS}$

R4$^{!}_{\mathcal{CS}}$

$$\overline{! \cdots !!c : \cdots !!c :!c : c : A},$$

where $c : A \in \mathcal{CS}$

$JL_{\mathcal{CS}} = JL_{\emptyset} +$ R4$_{\mathcal{CS}}$, if JL includes positive introspection and
$JL_{\mathcal{CS}} = JL_{\emptyset} +$ R4$^{!}_{\mathcal{CS}}$, otherwise.

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Fitting-Semantics (F-models)

$(W, R, V, \mathcal{A})$, where

- $W \neq \emptyset$ is the set of worlds
- R is a binary relation on W (accessibility relation),
- V assigns a subset of W to each propositional variable, p, and
- $\mathcal{A}$ assigns a subset of W to each pair of a justification term and a formula.

Fitting-Semantics (F-models)

The accessibility relation conditions

R must be...

- Reflexive if the logic includes the factivity axiom.
- Transitive if the logic includes the positive introspection axiom.
- Serial if the logic includes the consistency axiom.

Fitting-Semantics (F-models)

The admissible evidence function conditions

- Application closure: for any formulas ϕ, ψ and justification terms t, s,

$$\mathcal{A}(s, F \to G) \cap \mathcal{A}(t, F) \subseteq \mathcal{A}(s \cdot t, G).$$

- Sum closure: for any formula ϕ and justification terms t, s,

$$\mathcal{A}(t, \phi) \cup \mathcal{A}(s, \phi) \subseteq \mathcal{A}(t + s, \phi).$$

- Simplified $\mathcal{CS}$-closure: for any axiom A, constant c, such that $c : A \in \mathcal{CS}$,

$$\mathcal{A}(c, A) = W,$$

if positive introspection is included in the axioms, or

- $\mathcal{CS}$-closure: for any axiom A, constant c, such that $c : A \in \mathcal{CS}$,

$$\mathcal{A}(\underbrace{!! \cdots !}_{n+1} c, \underbrace{! \cdots !}_{n} c \cdots : c : A) = \mathcal{A}(c, A) = W,$$

otherwise.

Fitting-Semantics (F-models)

The admissible evidence function conditions

- Positive introspection closure: for any formula ϕ and justification term t,

$$\mathcal{A}(t,\phi) \subseteq \mathcal{A}(!t, t:\phi),$$

  if the logic includes positive introspection.
- Monotonicity: for any formula ϕ, justification term t and $a, b \in W$, if aRb and $a \in \mathcal{A}(t,\phi)$, then $b \in \mathcal{A}(t,\phi)$, if the consistency axiom is included.

Fitting-Semantics (F-models)

Truth is defined in the following way

Given a state a:

- $M, a \not\models \bot$.
- If p is a propositional variable, then $\mathcal{M}, a \models p$ iff $a \in V(p)$
- If ϕ, ψ are formulas, then $\mathcal{M}, a \models \phi \rightarrow \psi$ if and only if $M, a \models \psi$, or $\mathcal{M}, a \not\models \phi$.
- If ϕ is a formula and t a term, then $\mathcal{M}, a \models t : \phi$ if and only if $a \in \mathcal{A}(t, \phi)$ and for all $b \in W$, if aRb, then $\mathcal{M}, b \models \phi$.

$\mathsf{J}_{\mathcal{CS}}$, $\mathsf{JD}_{\mathcal{CS}}$, $\mathsf{JT}_{\mathcal{CS}}$, $\mathsf{J4}_{\mathcal{CS}}$, $\mathsf{JD4}_{\mathcal{CS}}$, $\mathsf{LP}_{\mathcal{CS}}$ are sound and complete w.r.t. their F-models. For $\mathsf{JD}_{\mathcal{CS}}$, $\mathsf{JD4}_{\mathcal{CS}}$, $\mathcal{CS}$ must be axiomatically appropriate.

M(Mkrtychev) - models

J, JD, JT, J4, JD4, LP are also sound and complete with respect to their M-models.

M-models are F-models of only one state, with one extra condition in the case of JD and JD4:

Consistent Evidence Condition: $\mathcal{A}(t, \perp) = \emptyset$

Since there is only one world, we may consider $\mathcal{A}(t, \phi) = \mathit{true}$, or just $\mathcal{A}(t, \phi)$ to be another way to say $\mathcal{A}(t, \phi) = W$. Similarly for V.

The $*$-calculi

$*$-expressions are expressions of the form $*(t, \phi)$, where t is a justification term and ϕ is a formula. On these expressions and given some constant specification, $\mathcal{CS}$, we can define the $*$-calculi. The $*_{\mathcal{CS}}$-calculus:

Axioms: $*(!! \cdots !!c, ! \cdots !!c : \cdots :!c : c : A)$,
where $c : A \in \mathcal{CS}$

$$\frac{*(s, \phi \to \psi) \qquad *(t, \phi)}{*(s \cdot t, \psi)}$$

$$\frac{*(t, \phi)}{*(s + t, \phi)} \qquad \frac{*(s, \phi)}{*(s + t, \phi)}$$

The $*$-calculi

And the $*!_{\mathcal{CS}}$-calculus:

Axioms: $*(c, A)$,
where $c : A \in \mathcal{CS}$, or

$$\frac{*(s, \phi \rightarrow \psi) \qquad *(t, \phi)}{*(s \cdot t, \psi)}$$

$$\frac{*(t, \phi)}{*(s + t, \phi)} \qquad \frac{*(s, \phi)}{*(s + t, \phi)}$$

$$\frac{*(t, \phi)}{*(!t, t : \phi)}$$

The $*$-calculi and why are they of interest here

Theorem (Mkrtychev, 1997)

Let $\mathcal{JL}$ be one of the justification logics mentioned above. For a set of $*$-expressions, $\mathcal{B}^*$, define $\mathcal{A} : Tm \times \mathcal{JL} \longrightarrow \{True, False\}$ s.t. for any term t and any formula ϕ,

$$\mathcal{A}(t, \phi) = true \iff \mathcal{B}^* \vdash_{*\mathcal{CS}} *(t, \phi).$$

Then, if there exists an admissible evidence function $\mathcal{A}_{ad}$ s.t. $\mathcal{B}^* \subseteq \{*(t, \phi) | \mathcal{A}_{ad}(t, \phi) = true\}$, then $\mathcal{A}$ is a legitimate admissible evidence function and for any term t and any formula ϕ,

$$\mathcal{A}(t, \phi) = true \implies \mathcal{A}_{ad}(t, \phi) = true.$$

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The $*$-calculi and why are they of interest here

Theorem (Krupski 2003, Kuznets 2008)

Let $\mathcal{CS}$ be a schematic constant specification in NP. Then,

1. There exists a non-deterministic algorithm that runs in polynomial time and determines, given a finite set S of $*$-expressions, a formula ϕ and a term t, whether

$$S \vdash_{*\mathcal{CS}} *(t, \phi)$$

2. There exists a non-deterministic algorithm that runs in polynomial time and determines, given a finite set S of $*$-expressions, a formula ϕ and a term t, whether

$$S \vdash_{*!\mathcal{CS}} *(t, \phi)$$

The Problem we are trying to solve

For justification logic $JL_{\mathcal{CS}}$ and efficiently decidable constant specification ($\mathcal{CS} \in P$, or NP), given a formula ϕ, decide whether $JL_{\mathcal{CS}} \vdash \phi$ or not.

Assumptions will be made on the properties of the constant specification.

How will upper bounds be established?

The cases of J, JT, J4, LP

- The compact character of M-models will be used.
- A tableau-like procedure will be used.
- M-models have two parts: a valuation and a possible evidence function.
- The procedure will have two parts.

Informally,

- Given a formula ψ, (non-determimistically) produce a branch of a tableau, starting with $F\ \psi$. As is usually the case, the branch will try to describe a countermodel of ψ.
- The branch will include not only formulas of the language, but also $*$-expressions.
- A prefixed $*$-expression of the form $T\ \ *(t, \phi)$ will correspond to $\mathcal{A}(t, \phi) = \textit{true}$ in the model.
- We can then check whether the $*$-calculus produces a negative $*$ expression from the positive ones in the tableau. We know this check fails if and only if there is some legitimate admissible evidence function that satisfies all positive $*$-expressions and no negative ones.

Upper bounds for Justification Logics

Theorem (Kuznets, 2000)

$J_{\mathcal{CS}}, JT_{\mathcal{CS}}, J4_{\mathcal{CS}}, LP_{\mathcal{CS}} \in \Pi_2^p$, for $\mathcal{CS}$ decidable almost schematic.

Theorem (Kuznets 2008)

$JD_{\mathcal{CS}}$ with a decidable, almost schematic and axiomatically appropriate $\mathcal{CS}$ is in Π_2^p.

It is easier to show that the corresponding satisfiability problem for these logics in in Σ_2^p.

The procedure used constructs non-deterministically a tableau branch. The non-propositional rules are the following. The first two are for logics $J_{\mathcal{CS}}$ and $J4_{\mathcal{CS}}$. The following two are used for $JT_{\mathcal{CS}}$ and $LP_{\mathcal{CS}}$.

$$\frac{T\ \ s:\psi}{T\ \ *(s,\psi)} \qquad \frac{F\ \ s:\psi}{F\ \ *(s,\psi)}$$

$$\frac{T\ \ s:\psi}{\begin{array}{c} T\ \ \psi \\ T\ \ *(s,\psi) \end{array}} \qquad \frac{F\ \ s:\psi}{F\ \ *(s,\psi) \mid F\ \ \psi}$$

If the branch is closed, reject. If not, check if for some $*$-expression produced, of the form $F\ \ *(t,R)$, $X \vdash_{*\mathcal{CS}} *(t,R)$, where X is the set of all positively prefixed $*$-expressions in the branch. If yes, then reject. Otherwise, accept.

Upper bounds for Justification Logics

The case of JD

The procedure for $JD_{\mathcal{CS}}$ is similar, but with few changes necessary, to account for the consistent evidence condition. New, numerical prefixes will be used to represent different models: we must make sure that the term-prefixed formulas are consistent - in other words that the set $\{\psi | \mathcal{M} \models t : \psi \text{ for some } t\}$ is satisfiable. Instead of the tableau rules introduced above, the following ones are used:

$$\frac{n\ T\ s : \psi}{\begin{array}{c} n\ T\ *(s, \psi) \\ n+1\ T\ \psi \end{array}} \qquad \frac{n\ F\ s : \psi}{n\ F\ *(s, \psi)}$$

Upper bounds for Justification Logics

The case of JD - an observation

Although the procedure for JD as it was originally formalized is based on M-models, the tableau construction seems to describe something that looks like an F-model.

The advantages and disadvantages of F- and M-models when the evidence is consistent

- General F-models are not convenient. Having many states does not help.
- This is why we consider M-models when studying the complexity of justification logics.
- There is only one world and the conditions for the admissible evidence are simple to check.
- Except in the case of JD and JD4. The consistent evidence condition ($\neg\mathcal{A}(t, \perp)$) is hard to check.
- In the case of JD, this consistency is checked by trying to satisfy the term-prefixed formulas with another model. The end result is an F-model, where each state has only one successor.
- Things look a little more complicated for JD4. However, a simple constraint on the Fitting semantics of this logic will provide a solution.

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- Things look a little more complicated for JD4. However, a simple constraint on the Fitting semantics of this logic will provide a solution.

$\mathrm{JD4}_{\mathcal{CS}}$ is sound and complete w.r.t. its F-models, for an axiomatically appropriate constant specification. Additionally, it is complete w.r.t. its F-models that satisfy the following property.

Strong Evidence Property: $\mathcal{M}, a \models t : F$ if and only if $a \in \mathcal{A}(t, F)$

It is useful for later on, given an F-type admissible evidence function $\mathcal{A}$ and a world u to define $\mathcal{A}_u$ to be the set $\{(t, \phi) | u \in \mathcal{A}(t, \phi)\}$.

The case of JD4

Small models - but not too small

Proposition

A formula is $\mathsf{JD4}_{\mathcal{CS}}$-satisfiable, where $\mathcal{CS}$ is axiomatically appropriate, if and only if it is satisfiable by an F-model $\mathcal{M} = (W, R, V, \mathcal{A})$ for $\mathsf{JD4}_{\mathcal{CS}}$ that additionally satisfies the following properties:

- W has exactly two elements, a, b.
- $R = \{(a, b), (b, b)\}$

This proposition will be very convenient, as we have reduced the number of worlds to just two, while avoiding the consistent evidence condition.
The 'if' direction is immediately apparent.

Proof

The 'only if' direction

Suppose ϕ is satisfied by $\mathcal{M}^* = (W, R, V', \mathcal{A}')$ at world a.

- There is an infinite sequence of elements of W, $\alpha = (a_i)_{i \in \mathbb{N}}$, such that $a_0 = a$, $i < j \Rightarrow a_i R a_j$ & $\mathcal{A}'_{a_i} \subseteq \mathcal{A}'_{a_j}$. (By Seriality, Transitivity and Monotonicity)
- For any $t : \psi$, there is at most one $j \in \mathbb{N}$, $\mathcal{M}^*, a_j \not\models t : \psi \rightarrow \psi$.
- Let b be a term of α, where $\mathcal{M}^*, b \models t : \psi \rightarrow \psi$ for any $t : \psi$, subformula of ϕ.
- $\mathcal{M}$ will be the model $(\{a, b\}, \{(a, b), (b, b)\}, V, \mathcal{A})$, where $V, \mathcal{A}$ agree on a, b with $V', \mathcal{A}'$.

Proof

$\mathcal{M}, a \models \phi$

More precisely, $\mathcal{M}, a \models \psi$ iff $\mathcal{M}^*, a \models \psi$, for any ψ, subformula of ϕ (by induction on ψ).

- The propositional cases are easy.
- For any χ, subformula of ϕ, $\mathcal{M}^*, b \models \chi$ iff $\mathcal{M}, b \models \chi$.
- So, assuming $\psi = t : \chi$,

$$M^*, a \models \psi$$

$$\Updownarrow$$

$$a \in \mathcal{A}'(t, \chi) \;\&\; M^*, b \models \chi$$

$$\Updownarrow$$

$$a \in \mathcal{A}(t, \chi) \;\&\; M, b \models \chi$$

$$\Updownarrow$$

$$M, a \models \psi$$

The upper bound for JD4

Using what we just proved, we can design an algorithm to establish that:

Proposition

$\mathsf{JD4}_{\mathcal{CS}}$ is in Π_2^p, for any axiomatically appropriate, schematic and efficiently decidable $\mathcal{CS}$.

The algorithm for the JD4 case

The algorithm is almost identical to what was used in all the other cases. One difference is that the formulas now have T, F and a, b as prefixes. The non-propositional cases of the tableau procedure are covered by the following rules.

$$\frac{a\ T\ s : G}{\begin{array}{c} a\ T\ \ *(s, G) \\ b\ T\ s : G \end{array}}, \qquad \frac{b\ T\ s : G}{\begin{array}{c} b\ T\ \ *(s, G) \\ b\ T\ G \end{array}},$$

$$\frac{a\ F\ s : G}{a\ F\ \ *(s, G) \mid a\ F\ G}, \qquad \frac{b\ F\ s : G}{b\ F\ \ *(s, G) \mid b\ F\ G}$$

The algorithm for the JD4 case

If the branch is closed, reject. If not, let X_w be the set of all positive w-prefixed $*$-expressions in the branch, for any $w \in \{a, b\}$. Decide in nondeterministic polynomial time if for some $*$-expression produced, of the form $w\ F\ \ *(t, R)$, whether $X_w \vdash_{*\mathcal{CS}} *(t, R)$. If yes, then reject. Otherwise, accept.

Lower bounds

Theorem (Milnikel 2007, Buss, Kuznets 2009)

$JL_{\mathcal{CS}}$ is Π_2^p-hard, for any $JL \in \{$ J, JD, JT, J4, JD4, LP $\}$ and any axiomatically appropriate, and schematically injective $\mathcal{CS}$.

Finally,

Corollary (Milnikel 2007, Buss, Kuznets 2009)

$JL_{\mathcal{CS}}$ is Π_2^p-complete, for any $JL \in \{$ J, JD, JT, J4, JD4, LP $\}$ and any axiomatically appropriate, schematically injective and efficiently decidable $\mathcal{CS}$.

Thank you.

Finally,

Corollary (Milnikel 2007, Buss, Kuznets 2009)

$JL_{\mathcal{CS}}$ is Π_2^p-complete, for any $JL \in \{$ J, JD, JT, J4, JD4, LP $\}$ and any axiomatically appropriate, schematically injective and efficiently decidable $\mathcal{CS}$.

Thank you.